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On the stochastic Allen–Cahn equation on networks with multiplicative noise

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Abstract. We consider a system of stochastic Allen–Cahn equations on a finite network represented by a finite graph. On each edge in the graph a multiplicative Gaussian noise driven stochastic Allen–Cahn equation is given with possibly different potential barrier heights supplemented by a continuity condition and a Kirchhoff-type law in the vertices. Using the semigroup approach for stochastic evolution equations in Banach spaces we obtain existence and uniqueness of solutions with sample paths in the space of continuous functions on the graph. We also prove more precise space-time regularity of the solution.

Keywords: stochastic evolution equations, stochastic reaction-diffusion equations on networks, analytic semigroups, stochastic Allen–Cahn equation.

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1 Introduction

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1, \ldots, \mathsf{e}_m$ and n vertices $\mathsf{v}_1, \ldots, \mathsf{v}_n$. We normalize and parametrize the edges on the interval $[0, 1]$. We denote by $\Gamma(\mathsf{v}_i)$ the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \left\{ j \in \{1, \ldots, m\} : \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \right\}.$$

Denoting by $\Phi := (\phi_{ij})_{n \times m}$ the so-called incidence matrix of the graph G, see Subsection 2.1 for more details, we aim to analyse the existence, uniqueness and regularity of solutions of

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the problem

$$\begin{cases} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x)u_j(x,t) \\ \qquad\qquad + \beta_j^2 u(x,t) - u(x,t)^3 \\ \qquad\qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t\in(0,T],\ x\in(0,1),\ j=1,\dots,m, \\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t\in(0,T],\ \forall j,\ell\in\Gamma(\mathsf{v}_i),\ i=1,\dots,n, \\ [Mq(t)]_i = -\displaystyle\sum_{j=1}^m \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t\in(0,T],\ i=1,\dots,n, \\ u_j(0,x) = \mathsf{u}_j(x), & x\in[0,1],\ j=1,\dots,m, \end{cases} \tag{1.1}$$

where $\frac{\partial w_j}{\partial t}$ are independent space-time white noises. The reaction terms in (1.1) are classical Allen–Cahn nonlinearities $h_j(\eta) = -\eta^3 + \beta_j^2\eta$ with $\beta_j > 0$, $j = 1,\dots,m$. Note that $h_j = -H_j'$ where $H_j(\eta) = \frac{1}{4}(\eta^2 - \beta_j^2)^2$ is a double well potential for each j with potential barrier height $\beta_j^4/4$. The diffusion coefficients g_j are assumed to be locally Lipschitz continuous and of linear growth. The coefficients of the linear operator satisfy standard smoothness assumptions, see Subsection 2.1, the matrix M satisfies Assumptions 2.7 and μ_j, $j = 1,\dots,m$, are positive constants. The classical Allen–Cahn equation belongs to the class of phase field models and is a classical tool to model processes involving thin interface layers between almost homogeneous regions, see [3]. It is a particular case of a reaction-diffusion equation of bistable type and it can be used to study front propagations as in [7] . Effects due to, for example, thermal fluctuations of the system can be accounted for by adding a Wiener type noise in the equation, see [20].

While deterministic evolution equations on networks are well studied, see, [1,2,5,6,8–11,17, 18,25,29–31,34–38] which is, admittedly, a rather incomplete list, the study of their stochastic counterparts is surprisingly scarce despite their strong link to applications, see e.g. [12,13,44] and the references therein. In [12] additive Lévy noise is considered that is square integrable with drift being a cubic polynomial. In [14] multiplicative square integrable Lévy noise is considered but with globally Lipschitz drifts f_j and diffusion coefficients and with a small time dependent perturbation of the linear operator. Paper [13] treats the case when the noise is an additive fractional Brownian motion and the drift is zero. In [22] multiplicative Wiener perturbation is considered both on the edges and vertices with globally Lipschitz diffusion coefficient and zero drift and time-delayed boundary condition. Finally, in [21], the case of multiplicative Wiener noise is treated with bounded and globally Lipschitz continuous drift and diffusion coefficients and noise both on the edges and vertices.

In all these papers the semigroup approach is utilized in a Hilbert space setting and the only work that treats non-globally Lipschitz continuous drifts on the edges, similar to the ones considered here, is [12] but the noise is there additive and square-integrable. In this case, energy arguments are possible using the additive nature of the equation which does not carry over to the multiplicative case. Therefore, we use an entirely different tool set based on the semigroup approach for stochastic evolution equations in Banach spaces [39], or for the classical stochastic reaction-diffusion setting [32,33], see also, [15,16,19,41]. We are able to rewrite (1.1) in a form that fits into this framework. After establishing various embedding and isomorphy results of function spaces and interpolation spaces, we may use [33, Theorem 4.9] to prove our main existence and uniqueness result, Theorem 3.15, which guarantees existence and uniqueness of solutions with sample paths in the space of continuous functions on the graph, denoted by B in the paper (see Definition 3.4); that is, in the space of continuous functions that are continuous on the edges and also across the vertices. When the initial data is sufficiently regular, then Theorem 3.15 also yields certain space-time regularity of the solution.

The paper is organized as follows. In Section 2 we collect partially known semigroup results for the linear deterministic version of (1.1). In Subsection 3.1 we first recall an abstract result from [32, 33] regarding abstract stochastic Cauchy problems in Banach spaces. In order to utilize the abstract framework in our setting we prove various preparatory results in Subsection 3.2: embedding and isometry results are contained in Lemma 3.5, Lemma 3.6 and Corollary 3.7, and a semigroup result in Proposition 3.8. Subsection 3.3 contains our main results where we first consider the abstract stochastic Itô equation corresponding to a slightly more general version of (1.1). An existence and uniqueness result for the abstract stochastic Itô problem is contained in Theorem 3.13 followed by a space-time regularity result in Theorem 3.14. These are then applied to the Itô equation corresponding (1.1) to yield the main result of the paper, Theorem 3.15, concerning the existence, uniqueness and space-time regularity of the solution of (1.1).

2 Heat equation on a network

2.1 The system of equations

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ and n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$. We normalize and parametrize the edges on the interval $[0, 1]$.

The structure of the network is given by the $n \times m$ matrices $\Phi^+ := (\phi^+_{ij})$ and $\Phi^- := (\phi^-_{ij})$ defined by

$$\phi^+_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi^-_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, n$ and $j = 1, \dots m$. We denote by $\mathsf{e}_j(0)$ and $\mathsf{e}_j(1)$ the 0 and the 1 endpoint of the edge e_j, respectively. We refer to [30] for terminology. The $n \times m$ matrix $\Phi := (\phi_{ij})$ defined by

$$\Phi := \Phi^+ - \Phi^-$$

is known in graph theory as *incidence matrix* of the graph G. Further, let $\Gamma(\mathsf{v}_i)$ be the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \left\{ j \in \{1, \dots, m\} : \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \right\}.$$

For the sake of simplicity, we will denote the values of a continuous function defined on the (parameterized) edges of the graph, that is of

$$f = (f_1, \dots, f_m)^\top \in (C[0,1])^m \cong C\left([0,1], \mathbb{R}^m\right)$$

at 0 or 1 by $f_j(\mathsf{v}_i)$ if $\mathsf{e}_j(0) = \mathsf{v}_i$ or $\mathsf{e}_j(1) = \mathsf{v}_i$, respectively, and $f_j(\mathsf{v}_i) := 0$ otherwise, for $j = 1, \dots, m$.

We start with the problem

$$\begin{cases} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x) u_j(t,x), & t > 0,\ x \in (0,1),\ j = 1, \dots, m, & (a) \\ u_j(t, \mathsf{v}_i) = u_\ell(t, \mathsf{v}_i) =: q_i(t), & t > 0,\ \forall j, \ell \in \Gamma(\mathsf{v}_i),\ i = 1, \dots, n, & (b) \\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij} \mu_j c_j(\mathsf{v}_i) u_j'(t, \mathsf{v}_i), & t > 0,\ i = 1, \dots, n, & (c) \\ u_j(0,x) = \mathsf{u}_j(x), & x \in [0,1],\ j = 1, \dots, m & (d) \end{cases} \tag{2.1}$$

on the network. Note that $c_j(\cdot)$, $p_j(\cdot)$ and $u_j(t,\cdot)$ are functions on the edge e_j of the network, so that the right-hand side of (2.1a) reads in fact as

$$(c_j u_j')'(t,\cdot) = \frac{\partial}{\partial x}\left(c_j \frac{\partial}{\partial x} u_j\right)(t,\cdot) - p_j(\cdot) u_j(t,\cdot), \qquad t \geq 0,\ j = 1,\ldots,m.$$

The functions $c_1,\ldots,c_m$ are (variable) diffusion coefficients or conductances, and we assume that

$$0 < c_j \in C^1[0,1], \qquad j = 1,\ldots,m.$$

The functions $p_1,\ldots,p_m$ are nonnegative, continuous functions, hence

$$0 \leq p_j \in C[0,1], \qquad j = 1,\ldots,m.$$

Equation (2.1b) represents the continuity of the values attained by the system at the vertices in each time instant, and we denote by $q_i(t)$ the common functions values in the vertice i, for $i = 1,\ldots,n$ and $t > 0$.

In (2.1c), $M := \left(b_{ij}\right)_{n\times n}$ is a matrix satisfying the following

Assumption 2.1. *The matrix $M = \left(b_{ij}\right)_{n\times n}$ is real, symmetric and negative semidefinite, $M \not\equiv 0$.*

On the left-hand-side, $[Mq(t)]_i$ denotes the ith coordinate of the vector $Mq(t)$. On the right-hand-side, the coefficients

$$0 < \mu_j, \quad j = 1,\ldots,m$$

are strictly positive constants that influence the distribution of impulse happening in the ramification nodes according to the Kirchhoff-type law (2.1c).

We now introduce the $n \times m$ *weighted incidence matrices*

$$\Phi_w^+ := (\omega_{ij}^+) \quad \text{and} \quad \Phi_w^- := (\omega_{ij}^-)$$

with entries

$$\omega_{ij}^+ := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \omega_{ij}^- := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise.} \end{cases}$$

With these notations, the Kirchhoff law (2.1c) becomes

$$Mq(t) = -\Phi_w^+ u'(t,0) + \Phi_w^- u'(t,1), \qquad t \geq 0. \tag{2.2}$$

In equation (2.1d) we pose the initial conditions on the edges.

2.2 Spaces and operators

We are now in the position to rewrite our system in form of an abstract Cauchy problem, following the concept of [31]. First we consider the (real) Hilbert space

$$E_2 := \prod_{j=1}^{m} L^2(0,1;\mu_j dx) \tag{2.3}$$

as the *state space* of the edges, endowed with the natural inner product

$$\langle u,v\rangle_{E_2} := \sum_{j=1}^{m} \int_0^1 u_j(x) v_j(x) \mu_j dx, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in E_2.$$

Observe that E_2 is isomorphic to $\left(L^2(0,1)\right)^m$ with equivalence of norms.

We further need the *boundary space* $\mathbb{R}^n$ of the vertices. According to (2.1b) we will consider such functions on the edges of the graph those values coincide in each vertex. Therefore we introduce the *boundary value operator*

$$L\colon\ (C[0,1])^m \subset E_2 \to \mathbb{R}^n$$

with

$$\begin{aligned} D(L) &= \left\{u \in (C[0,1])^m : u_j(\mathsf{v}_i) = u_\ell(\mathsf{v}_i),\ \forall j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n\right\};\\ Lu &:= (q_1,\dots,q_n)^\top \in \mathbb{R}^n, \qquad q_i = u_j(\mathsf{v}_i) \quad \text{for some } j \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n. \end{aligned} \tag{2.4}$$

The condition $u(t,\cdot) \in D(L)$ for each $t > 0$ means that (2.1b) is for the function $u(\cdot,\cdot)$ satisfied.

On E_2 we define the operator

$$A_{\max} := \begin{pmatrix} \frac{d}{dx}\left(c_1\frac{d}{dx}\right) - p_1 & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m\frac{d}{dx}\right) - p_m \end{pmatrix} \tag{2.5}$$

with domain

$$D(A_{\max}) := \left(H^2(0,1)\right)^m \cap D(L). \tag{2.6}$$

This operator can be regarded as *maximal* since no other boundary condition except continuity is supposed for the functions in its domain.

We further define the so called *feedback operator* acting on $D(A_{\max})$ and having values in the boundary space $\mathbb{R}^n$ as

$$\begin{aligned} D(C) &= D(A_{\max});\\ Cu &:= -\Phi_w^+ u'(0) + \Phi_w^- u'(1), \end{aligned}$$

compare with (2.2).

With these notations, we can finally rewrite (2.1) in form of an abstract Cauchy problem. Define

$$\begin{aligned} A &:= A_{\max} \\ D(A) &:= \{u \in E_2 : u \in D(A_{\max}) \text{ and } MLu = Cu\}, \end{aligned} \tag{2.7}$$

see the definitions above. Using this, (2.1) becomes

$$\begin{cases} \dot{u}(t) = Au(t), & t > 0,\\ u(0) = \mathsf{u}, \end{cases} \tag{2.8}$$

with $\mathsf{u} = (\mathsf{u}_1,\dots,\mathsf{u}_m)^\top$.

2.3 Well-posedness of the abstract Cauchy problem

To prove well-posedness of (2.8) we define a bilinear form on the Hilbert space E_2 with domain

$$D(\mathfrak{a}) = V := \left(H^1(0,1)\right)^m \cap D(L). \tag{2.9}$$

as

$$\mathfrak{a}(u,v) := \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) u_j'(x) v_j'(x) dx + \sum_{j=1}^{m} \int_0^1 \mu_j p_j(x) u_j(x) v_j(x) dx - \langle Mq, r \rangle_{\mathbb{R}^n}, \tag{2.10}$$

where $Lu = q$ and $Lv = r$.

The next definition can be found e.g. in [40, Section 1.2.3].

Definition 2.2. From the form $\mathfrak{a}$ – using the Riesz representation theorem – we can obtain a unique operator $(B, D(B))$ in the following way:

$$\begin{aligned} D(B) &:= \{ u \in V : \exists v \in E_2 \text{ s.t. } \mathfrak{a}(u,\varphi) = \langle v, \varphi \rangle_{E_2} \; \forall \varphi \in V \}, \\ Bu &:= -v. \end{aligned}$$

We say that the operator $(B, D(B))$ is *associated with the form* $\mathfrak{a}$.

In the following, we will claim that the operator associated with the form $\mathfrak{a}$ is $(A, D(A))$. Furthermore, we will state results regarding how the properties of $\mathfrak{a}$ and the matrix M carry on the properties of the operator A, obtaining the well-posedness of the abstract Cauchy-problem (2.8) on E_2 and even on L^p-spaces of the edges. The proofs of these statements combine techniques of [36] (where no p_j's on the right-hand-side of (2.1b) are considered) and techniques of [38] (where p_j's are considered for the heat equation but the matrix M is diagonal).

Proposition 2.3. *The operator associated to the form $\mathfrak{a}$ (2.9)–(2.10) is $(A, D(A))$ in (2.7).*

Proof. We can proceed similarly as in the proofs of [36, Lemma 3.4] and [38, Lemma 3.3]. □

Proposition 2.4. *The form $\mathfrak{a}$ is densely defined, continuous, closed and accretive, hence $(A, D(A))$ is densely defined, dissipative and sectorial. Furthermore, $\mathfrak{a}$ is symmetric, hence the operator $(A, D(A))$ is self-adjoint.*

Proof. The first three properties of $\mathfrak{a}$ (densely defined, continuous and closed) follow analogous to the proof of [38, Lemma 3.2]. Since M is dissipative (that is, negative semidefinite), and $p_j \geq 0$, $j = 1, \ldots, m$, the form $\mathfrak{a}$ is accretive, see the proofs of [36, Proposition 3.2] and [38, Lemma 3.2]. The symmetricity of $\mathfrak{a}$ follows from the fact that M is real and symmetric, see the proof of [36, Corollary 3.3]. The properties of A follow now by [40, Proposition 1.24, 1.51, Theorem 1.52]. □

As a corollary we obtain well-posedness of (2.8).

Proposition 2.5. *Assuming Assumption 2.1 on the matrix M, the operator $(A, D(A))$ defined in (2.7) generates a C_0 analytic, compact semigroup of contractions $(T_2(t))_{t\geq 0}$ on E_2. Hence, the abstract Cauchy problem (2.8) is well-posed on E_2.*

Proof. The claim follows from Proposition 2.4 and the fact that $(A, D(A))$ is resolvent compact. This is true since V is densely and compactly embedded in E_2 by the Rellich–Khondrakov Theorem, and we can use [24, Theorem 1.2.1]. □

In the following we will extend the semigroup $(T_2(t))_{t\geq 0}$ on L^p-spaces. To this end we define

$$E_p := \prod_{j=1}^{m} L^p(0,1;\mu_j dx), \qquad p \in [1,\infty]$$

and

$$\|u\|_{E_p}^p := \sum_{j=1}^{m} \|u_j\|_{L^p(0,1;\mu_j dx)}^p, \qquad u \in E_p, \quad p \in [1,\infty),$$
$$\|u\|_{E_\infty} := \max_{j=1,\dots,m} \|u_j\|_{L^\infty(0,1)}, \qquad u \in E_\infty.$$

We can characterize features of the semigroup $(T_2(t))_{t\geq 0}$ by those of $(\mathrm{e}^{tM})_{t\geq 0}$, the semigroup generated by the matrix M – hence, by properties of M. In particular, the following holds.

Proposition 2.6. *The semigroup $(T_2(t))_{t\geq 0}$ on E_2 associated with $\mathfrak{a}$ enjoys the following properties:*

- *$(T_2(t))_{t\geq 0}$ is positive if and only if the matrix M has positive off-diagonal – that is, if it generates a positive matrix semigroup $(\mathrm{e}^{tM})_{t\geq 0}$;*
- *Since M is negative semidefinite, the semigroup $(T_2(t))_{t\geq 0}$ is contractive on E_∞ if and only if*

$$b_{ii} + \sum_{k\neq i} |b_{ik}| \leq 0, \qquad i = 1,\dots,n,$$

that is $(\mathrm{e}^{tM})_{t\geq 0}$ is ℓ^∞-contractive.

Proof. It follows using analogous techniques as in the proof of [36, Theorem 3.5] and [38, Lemma 4.1, Proposition 5.3] □

To obtain the desired extension of the semigroup on L^p-spaces, we assume the following on the matrix M.

Assumption 2.7. *For the matrix $M = (b_{ij})_{n\times n}$ we assume the following properties:*

1. *M satisfies Assumption 2.1;*
2. *For $i \neq k$, $b_{ik} \geq 0$, that is, M has positive off-diagonal;*
3. $$\sum_{k\neq i} b_{ik} \leq -b_{ii}, \quad i = 1,\dots,n,$$
   that is, the matrix is diagonally dominant.

Proposition 2.8. *If M satisfies Assumptions 2.7 then the semigroup $(T_2(t))_{t\geq 0}$ extends to a family of compact, contractive, positive one-parameter semigroups $(T_p(t))_{t\geq 0}$ on E_p, $1 \leq p \leq \infty$. Such semigroups are strongly continuous if $p \in [1,\infty)$, and analytic of angle $\frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}}$ for $p \in (1,\infty)$.*

Moreover, the spectrum of A_p is independent of p, where A_p denotes the generator of $(T_p(t))_{t\geq 0}$, $1 \leq p \leq \infty$.

Proof. It follows by [4, Section 7.2] as in [36, Theorem 4.1] and [38, Corollary 5.6]. □

We also can prove that the generators of the semigroups in the spaces E_p, $1 \leq p \leq \infty$ have in fact the same form as in E_2, with appropriate domain.

Lemma 2.9. *For all $p \in [1,\infty]$ the generator A_p of the semigroup $(T_p(t))_{t\geq 0}$ is given by the operator defined in (2.5) with domain*

$$D(A_p) = \left\{ u \in \prod_{j=1}^{m} W^{2,p}(0,1;\mu_j dx) \cap D(L) : MLu = Cu \right\}. \tag{2.11}$$

In particular, A_p has compact resolvent for $p \in [1,\infty]$.

Proof. See [36, Proposition 4.6] and [38, Lemma 5.7]. □

As a summary we obtain the following theorem.

Theorem 2.10. *The first order problem (2.1) is well-posed on E_p, $p \in [1,\infty)$, i.e., for all initial data* u $\in E_p$ *the problem (2.1) admits a unique mild solution that continuously depends on the initial data.*

3 The stochastic Allen–Cahn equation on networks

3.1 An abstract stochastic Cauchy problem

Let $(\Omega,\mathscr{F},\mathbb{P})$ is a complete probability space endowed with a right continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$. Let $(W_H(t))_{t\in[0,T]}$ be a cylindrical Wiener process, defined on $(\Omega,\mathscr{F},\mathbb{P})$, in some Hilbert space H with respect to the filtration $\mathbb{F}$; that is, $(W_H(t))_{t\in[0,T]}$ is $(\mathscr{F}_t)_{t\in[0,T]}$-adapted and for all $t > s$, $W_H(t) - W_H(s)$ is independent of $\mathscr{F}_s$. To be able to handle the stochastic Allen–Cahn equation on networks, first we cite a result of M. Kunze and J. van Neerven, regarding the following abstract equation

$$\begin{cases} dX(t) = [AX(t) + F(t,X(t))]dt + G(t,X(t))dW_H(t) \\ X(0) = \xi, \end{cases} \tag{SCP}$$

see [32, Section 3]. If we assume that $(A,D(A))$ generates a strongly continuous, analytic semigroup S on the Banach space E with $\|S(t)\| \leq Ke^{\omega t}$, $t \geq 0$ for some $K \geq 1$ and $\omega \in \mathbb{R}$, then for $\omega' > \omega$ the fractional powers $(\omega' - A)^\alpha$ are well-defined for all $\alpha \in (0,1)$. In particular, the fractional domain spaces

$$E^\alpha := D((\omega' - A)^\alpha), \quad \|v\|_\alpha := \|(\omega' - A)^\alpha v\|, \quad v \in D((\omega' - A)^\alpha) \tag{3.1}$$

are Banach spaces. It is well-known (see e.g. [26, §II.4–5.]), that up to equivalent norms, these spaces are independent of the choice of ω'.

For $\alpha \in (0,1)$ we define the extrapolation spaces $E^{-\alpha}$ as the completion of E under the norms $\|v\|_{-\alpha} := \|(\omega' - A)^{-\alpha} v\|$, $v \in E$. These spaces are independent of $\omega' > \omega$ up to an equivalent norm.

We fix $E^0 := E$.

Remark 3.1. If A is injective and $\omega = 0$ (hence, the semigroup S is bounded), then by [28, Chapter 6.2, Introduction] we can choose $\omega' = 0$. That is,

$$E^\alpha \cong D((-A)^\alpha), \qquad \alpha \in [0,1).$$

To obtain the desired result for the solution of (SCP), one has to impose the following assumptions for the mappings in (SCP). These are – in the first and third cases slightly simplified versions of – Assumptions (A1), (A5), (A4), (F'), (F'') and (G'') in [32]. Let B be a Banach space, $\|\cdot\|$ will denote $\|\cdot\|_B$. For $u \in B$ we define the *subdifferential of the norm at u* as the set

$$\partial\|u\| := \{u^* \in B^* : \|u^*\| = 1 \text{ and } \langle u, u^* \rangle = 1\} \tag{3.2}$$

which is not empty by the Hahn–Banach theorem. Furthermore, let E be a UMD Banach space of type 2.

Assumptions 3.2.

1. *$(A, D(A))$ is densely defined, closed and sectorial on E.*
2. *For some $0 \le \theta < \frac{1}{2}$ we have continuous, dense embeddings*

$$E^\theta \hookrightarrow B \hookrightarrow E.$$

3. *Let S be the strongly continuous analytic semigroup generated by $(A, D(A))$. Then S restricts to a strongly continuous contraction semigroup S^B on B, in particular, $A|_B$ is dissipative.*
4. *The map $F\colon [0, T] \times \Omega \times B \to B$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_F^{(r)}$ such that*

$$\|F(t, \omega, u) - F(t, \omega, v)\| \le L_F^{(r)} \|u - v\|$$

for all $\|u\|, \|v\| \le r$ and $(t, \omega) \in [0, T] \times \Omega$ and there exists a constant $C_{F,0} \ge 0$ such that

$$\|F(t, \omega, 0)\| \le C_{F,0}, \qquad t \in [0, T],\ \omega \in \Omega.$$

Moreover, for all $u \in B$ the map $(t, \omega) \mapsto F(t, \omega, u)$ is strongly measurable and adapted. Finally, for suitable constants $a, b \ge 0$ and $N \ge 1$ we have

$$\langle Au + F(t, u + v), u^* \rangle \le a(1 + \|v\|)^N + b\|u\|$$

for all $u \in D(A|_B)$, $v \in B$ and $u^* \in \partial\|u\|$, see (3.2).

5. *There exist constants a'', b'', $m' > 0$ such that the function $F\colon [0, T] \times \Omega \times B \to B$ satisfies*

$$\langle F(t, \omega, u + v) - F(t, \omega, v), u^* \rangle \le a''(1 + \|v\|)^{m'} - b''\|u\|^{m'}$$

for all $t \in [0, T]$, $\omega \in \Omega$, $u, v \in B$ and $u^* \in \partial\|u\|$, and

$$\|F(t, v)\| \le a''(1 + \|v\|)^{m'}$$

for all $v \in B$.

6. *Let $\gamma(H, E^{-\kappa_G})$ denote the space of γ-radonifying operators from H to $E^{-\kappa_G}$ for some $0 \le \kappa_G < \frac{1}{2}$, see e.g. [32, Section 3.1]. Then the map $G\colon [0, T] \times \Omega \times B \to \gamma(H, E^{-\kappa_G})$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_G^{(r)}$ such that*

$$\|G(t, \omega, u) - G(t, \omega, v)\|_{\gamma(H, E^{-\kappa_G})} \le L_G^{(r)} \|u - v\|$$

for all $\|u\|, \|v\| \le r$ *and* $(t,\omega) \in [0,T] \times \Omega$. *Moreover, for all* $u \in B$ *and* $h \in H$ *the map* $(t,\omega) \mapsto G(t,\omega,u)h$ *is strongly measurable and adapted.*

Finally, G is of linear growth, that is, for suitable constant c',

$$\|G(t,\omega,u)\|_{\gamma(H,E^{-\kappa_G})} \le c'(1+\|u\|)$$

for all $(t,\omega,u) \in [0,T] \times \Omega \times B$.

Recall that a *mild solution* of (SCP) is a solution of the following implicit equation

$$\begin{aligned} X(t) &= S(t)\xi + \int_0^t S(t-s)F(s,X(s))\,ds + \int_0^t S(t-s)G(s,X(s))\,dW_H(s) \\ &=: S(t)\xi + S * F(\cdot,X(\cdot))(t) + S \diamond G(\cdot,X(\cdot))(t) \end{aligned} \tag{3.3}$$

where

$$S * f(t) = \int_0^t S(t-s)f(s)\,ds$$

denotes the "usual" convolution, and

$$S \diamond g(t) = \int_0^t S(t-s)g(s)\,dW_H(s)$$

denotes the stochastic convolution with respect to W_H.

The result of Kunze and van Neerven that will be useful for our setting is the following. We note that this was first proved in [32, Theorem 4.9] but with a typo in the statement which was later corrected in the recent arXiv preprint [33, Theorem 4.9].

Theorem 3.3 ([33, Theorem 4.9])**.** *Suppose that Assumptions 3.2 hold and let* $2 < q < \infty$, $0 \le \theta < \frac{1}{2}$, $0 \le \kappa_G < \frac{1}{2}$ *satisfy*

$$\theta + \kappa_G < \frac{1}{2} - \frac{1}{q}.$$

Then for all $\xi \in L^q(\Omega,\mathscr{F}_0,\mathbb{P};B)$ *there exists a unique global mild solution*

$$X \in L^q(\Omega, C([0,T];B))$$

of (SCP)*. Moreover, for some constant* $C > 0$ *we have*

$$\mathbb{E}\|X\|^q_{C([0,T];B)} \le C \cdot (1 + \mathbb{E}\|\xi\|^q)\,.$$

3.2 Preparatory results

In order to apply the abstract result of Theorem 3.3 to the stochastic Allen–Cahn equation on a network we need to prove some preparatory results using the setting of Section 2.

On the edges of the graph G we will consider continuous functions that satisfy the continuity condition in the vertices, see Subsection 2.1. We will refer to such functions as *continuous functions on the graph G* and denote them by $C(G)$.

Definition 3.4. We define

$$C(G) := D(L),$$

see (2.4), which can be looked at as the Banach space of all continuous functions on the graph G, hence the norm on $C(G)$ can be defined as

$$\|u\|_{C(G)} = \max_{j=1,\dots,m} \sup_{[0,1]} |u_j|, \quad u \in C(G).$$

This space will play the role of the space B in our setting, hence we set

$$B := C(G) \text{ and } \|\cdot\|_{C(G)} := \|\cdot\|_B. \tag{3.4}$$

We will show that for θ big enough the continuous, dense embeddings

$$E_p^\theta \hookrightarrow B \hookrightarrow E_p$$

hold, where

$$E_p^\theta \text{ is defined for the operator } A_p \text{ on the Banach space } E_p \text{ as in (3.1).} \tag{3.5}$$

To do so, we first need a technical lemma, and define the maximal operator on E_p as

$$A_{p,\max} := \begin{pmatrix} \frac{d}{dx}\left(c_1 \frac{d}{dx}\right) - p_m & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m \frac{d}{dx}\right) - p_m \end{pmatrix} \tag{3.6}$$

with domain

$$D(A_{p,max}) := \left(\prod_{j=1}^m W^{2,p}(0,1;\mu_j dx)\right) \cap D(L), \tag{3.7}$$

see (2.5) (2.6) in E_2. Hence, the domain of $A_{p,\max}$ only contains the continuity condition in the nodes.

Furthermore, define

$$W_0(G) := \prod_{j=1}^m W_0^{2,p}(0,1;\mu_j dx), \tag{3.8}$$

where

$$W_0^{2,p}(0,1;\mu_j dx) = W^{2,p}(0,1;\mu_j dx) \cap W_0^{1,p}(0,1;\mu_j dx), \qquad j = 1,\dots,m.$$

That is, $W_0(G)$ contains such vectors of functions that are twice weakly differentiable on each edge and continuous on the graph with Dirichlet boundary conditions.

Lemma 3.5.

$$D(A_{p,max}) \cong W_0(G) \times \mathbb{R}^n,$$

where the isomorphism is taken for $D(A_{p,\max})$ equipped with the operator graph norm.

Proof. We will use the setting of [27] for $A = A_{p,\max}$, $X = E_p$ and the boundary operator $L : D(L) \subset E_p \to \mathbb{R}^n =: Y$. Denote

$$A_0 := A_{p,\max}|_{\ker L},$$

which is the operator (3.6) with Dirichlet boundary conditions. Hence, it is a generator on E_p. Clearly

$$D(A_0) = W_0(G) \tag{3.9}$$

holds.

We now choose $\lambda \in \rho(A_0)$. Using [27, Lemma 1.2] we have that

$$D(A_{p,\max}) = D(A_0) \oplus \ker(\lambda - A_{p,\max}).$$

Furthermore, the map

$$L\colon \ker(\lambda - A_{p,\max}) \to \mathbb{R}^n \tag{3.10}$$

is an onto isomorphism, having the inverse

$$D_\lambda := (L|_{\ker(\lambda - A_{p,\max})})^{-1}\colon \mathbb{R}^n \to \ker(\lambda - A_{p,\max})$$

called *Dirichlet-operator*, see [27, (1.14)]. By [27, (1.15)],

$$D_\lambda L\colon D(A_{p,\max}) \to \ker(\lambda - A_{p,\max})$$

is the projection in $D(A_{p,\max})$ onto $\ker(\lambda - A_{p,\max})$ along $D(A_0)$. Since $D_\lambda L$ is continuous, by the properties of the direct sum, see e.g. [42, Theorem 2.5], we obtain that

$$D(A_{p,\max}) \cong D(A_0) \times \ker(\lambda - A_{p,\max})$$

holds. Now using (3.9) and that (3.10) is an isomorphism, the claim follows. □

Lemma 3.6. *For the space B defined in* (3.4)

$$B \cong (C_0[0,1])^m \times \mathbb{R}^n$$

holds.

Proof. Let $u \in B$ arbitrary and $r := Lu \in \mathbb{R}^n$. We can define the unique $v^u \in B$ such that v^u_j is a first order polynomial for each $j = 1, \dots, m$ taking values

$$v^u_j(\mathsf{v}_i) = r_i, \quad \text{for } \mathsf{e}_j \in \Gamma(\mathsf{v}_i)\ j = 1, \dots, m,\ i = 1, \dots, n.$$

Then $Lv^u = r$ and

$$u - v^u \in (C_0[0,1])^m.$$

Denote

$$B_1 := \{v^u : u \in B\} \subset B$$

a closed subspace. Clearly,

$$(C_0[0,1])^m \cap B_1 = \{0_B\}$$

and if $u \in B$ then $u = (u - v^u) + v^u$ with $u - v^u \in (C_0[0,1])^m$ and $v^u \in B_1$. Hence

$$B = (C_0[0,1])^m \oplus B_1.$$

By the construction of v^u follows that since $L : B \to \mathbb{R}^n$ is onto,

$$L|_{B_1} \colon B_1 \to \mathbb{R}^n$$

is a bijection. The operator $L|_{B_1}$ is also bounded for the norm of B induced on B_1. Hence, by the open mapping theorem, it is an isomorphism. Denoting its inverse by

$$L_1 := (L|_{B_1})^{-1} : \mathbb{R}^n \to B_1,$$

we obtain that

$$L_1 L\colon B \to B_1$$

is the continuous projection from B onto B_1 along $(C_0[0,1])^m$. Hence, we can use [42, Theorem 2.5] and obtain

$$B \cong (C_0[0,1])^m \times \mathbb{R}^n.$$

□

Corollary 3.7. *Let E_p^θ defined in (3.5). If $\theta > \frac{1}{2p}$ then the following continuous, dense embeddings are satisfied:*

$$E_p^\theta \hookrightarrow B \hookrightarrow E_p. \tag{3.11}$$

Proof. We know that $(A_p, D(A_p))$ is sectorial and maximal dissipative, hence it is injective and generates a contractive semigroup. By Remark 3.1 we have that

$$E_p^\theta \cong D((-A_p)^\theta)$$

for $\theta \in [0,1)$. It follows from [4, Theorem in §5.3.5] and [4, Theorem in §4.4.10] that for the complex interpolation spaces

$$D((-A_p)^\theta) \cong [D(-A_p), E_p]_\theta,$$

hence

$$E_p^\theta \cong [D(-A_p), E_p]_\theta$$

holds with equivalence of norms. Defining $(A_{p,\max}, D(A_{p,\max}))$ as in (3.6), (3.7) we have that

$$D(A_p) \hookrightarrow D(A_{p,\max})$$

holds. Hence

$$E_p^\theta \hookrightarrow \left[D(-A_{p,\max}), E_p\right]_\theta. \tag{3.12}$$

By Lemma 3.5,

$$D(-A_{p,\max}) \cong W_0(G) \times \mathbb{R}^n \tag{3.13}$$

holds, where $W_0(G)$ is defined in (3.8). Since $E_p \cong E_p \times \{0_{\mathbb{R}^n}\}$, using general interpolation theory, see e.g. [43, Section 4.3.3], we have that for $\theta > \frac{1}{2p}$

$$\left[W_0(G) \times \mathbb{R}^n, E_p \times \{0_{\mathbb{R}^n}\}\right]_\theta \hookrightarrow \left(\prod_{j=1}^m W_0^{2\theta,p}(0,1;\mu_j dx)\right) \times \mathbb{R}^n.$$

Thus, by (3.12) and (3.13)

$$E_p^\theta \hookrightarrow \left(\prod_{j=1}^m W_0^{2\theta,p}(0,1;\mu_j dx)\right) \times \mathbb{R}^n \tag{3.14}$$

holds. Hence,

$$E_p^\theta \hookrightarrow (C_0[0,1])^m \times \mathbb{R}^n \tag{3.15}$$

is true. Applying Lemma 3.6 we obtain that for $\theta > \frac{1}{2p}$

$$E_p^\theta \hookrightarrow B \tag{3.16}$$

is satisfied. Using Lemma 3.6 again, we have $B \hookrightarrow E_p$, and the claim follows. □

In the following we will prove that the part of the operator $(A_p, D(A_p))$ in B is the generator of a strongly continuous semigroup on B. First notice that by the form (2.11) of $D(A_p)$ and by (3.11)

$$D(A_p) \subset B \hookrightarrow E_p \tag{3.17}$$

holds.

Proposition 3.8. *The part of $(A_p, D(A_p))$ in B generates a positive strongly continuous semigroup of contractions on B.*

Proof. 1. We first prove that the semigroup $(T_p(t))_{t\geq 0}$ leaves B invariant. We take $u \in B$ and use that $(T_p(t))_{t\geq 0}$ is analytic on E_p (see Proposition 2.8). Hence, $T_p(t)u \in D(A_p)$. By (3.17) also

$$T_p(t)u \in B$$

holds.

2. In the next step we prove that $(T_p(t)|_B)_{t\geq 0}$ is a strongly continuous semigroup. By [26, Proposition I.5.3], it is enough to prove that there exist $K > 0$ and $\delta > 0$ and a dense subspace $D \subset B$ such that

(a) $\|T_p(t)\|_B \leq K$ for all $t \in [0, \delta]$, and

(b) $\lim_{t\downarrow 0} T_p(t)u = u$ for all $u \in D$.

To verify (a), we obtain by Proposition 2.8 that for $u \in B$

$$\|T_p(t)u\|_B = \|T_p(t)u\|_{E_\infty} = \|T_\infty(t)u\|_{E_\infty} \leq \|u\|_{E_\infty} = \|u\|_B,$$

hence

$$\|T_p(t)\|_B \leq 1 =: K, \quad t \geq 0.$$

To prove (b) take $\frac{1}{2p} < \theta < \frac{1}{2}$ arbitrary. By (3.11) we have that

$$D := E_p^\theta \hookrightarrow B$$

with dense, continuous embedding. Hence, there exists $C > 0$ such that for $u \in D$,

$$\begin{aligned} \|T_p(t)u - u\|_B &\leq C \cdot \|T_p(t)u - u\|_{E_p^\theta} \\ &= C \cdot \|T_p(t)(-A_p)^\theta u - (-A_p)^\theta u\|_{E_p} \to 0, \quad t \downarrow 0. \end{aligned}$$

Summarizing 1. and 2., and using (3.17), we can apply [26, Proposition in Section II.2.3] for $(A_p, D(A_p))$ and $Y = B$, and obtain that the part of $(A_p, D(A_p))$ in B generates a positive strongly continuous semigroup of contractions on B. □

Corollary 3.9. *The first order problem (2.1) is well-posed on B, i.e., for all initial data $\mathsf{u} \in B$ the problem (2.1) admits a unique mild solution that continuously depends on the initial data.*

3.3 Main results

In this subsection we first apply the above results to the following stochastic evolution equation, based on (2.1). This corresponds to a slightly more general version of (1.1), see (3.33) later.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space endowed with a right-continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ for some $T > 0$ given. We consider the problem

$$\left\{\begin{array}{lll} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x)u_j(t,x) & & \\ \qquad + f_j(t,x,u_j(t,x)) & & \\ \qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t \in (0,T],\ x \in (0,1),\ j = 1,\dots,m, & (a) \\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t \in (0,T],\ \forall j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n, & (b) \\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t \in (0,T],\ i = 1,\dots,n, & (c) \\ u_j(0,x) = \mathsf{u}_j(x), & x \in [0,1],\ j = 1,\dots,m, & (d) \end{array}\right. \tag{3.18}$$

where $\frac{\partial w_j}{\partial t}$, $j = 1, \dots, m$, are independent space-time white noises on $[0,1]$; written as formal derivatives of independent cylindrical Wiener-processes $(w_j(t))_{t\in[0,T]}$, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, in the Hilbert space $L^2(0,1;\mu_j dx)$ with respect to the filtration $\mathbb{F}$.

The functions $f_j\colon [0,T] \times \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are polynomials of the form

$$f_j(t,\omega,x,\eta) = -a_{j,2k+1}(t,\omega,x)\eta^{2k+1} + \sum_{l=0}^{2k} a_{j,l}(t,\omega,x)\eta^l, \quad \eta \in \mathbb{R},\, j = 1,\dots,m \tag{3.19}$$

for some fixed integer k. For the coefficients we assume that there are constants $0 < c \leq C < \infty$ such that

$$c \leq a_{j,2k+1}(t,\omega,x) \leq C, \ \left|a_{j,l}(t,\omega,x)\right| \leq C, \text{ for all } j = 1,\dots,m, \ l = 0,2,\dots,2k,$$

for all $x \in [0,1]$, $t \in [0,T]$ and almost all $\omega \in \Omega$, see [32, Example 4.2]. The coefficients $a_{j,l}\colon [0,T] \times \Omega \times [0,1] \to \mathbb{R}$ are jointly measurable and adapted in the sense that for each j and l and for each $t \in [0,T]$, the function $a_{j,l}(t,\cdot)$ is $\mathscr{F}_t \otimes \mathcal{B}_{[0,1]}$-measurable, where $\mathcal{B}_{[0,1]}$ denotes the sigma-algebra of the Borel sets on $[0,1]$.

We further assume a technical assumption regarding the graph structure that will play and important role in our setting.

Assumption 3.10. *For the coefficients in* (3.19) *we assume that*

$$\left(a_{1,l}(t,\omega,\cdot),\dots,a_{m,l}(t,\omega,\cdot)\right)^\top \in B \text{ for all } l = 1,\dots,2k+1,$$

$t \in [0,T]$ *and almost all* $\omega \in \Omega$.

Remark 3.11. If the coefficients in (3.19) do not depend on j – that is, they are the same on different edges –, and satisfy

$$a_l(t,\omega,\cdot) = a_{j,l}(t,\omega,\cdot) \in C[0,1], \quad t \in [0,T], \omega \in \Omega, \quad j = 1,\dots m, \ l = 1,\dots,2k+1$$

and

$$a_l(t,\omega,0) = a_l(t,\omega,1), \text{ for all } l = 1,\dots 2k+1,$$

then Assumption 3.10 is fulfilled. This is the case e.g. if $a_l's$ are constant (not depending on x).

For the functions g_j we assume

$$\begin{aligned} &g_j\colon [0,T] \times \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}, \quad j = 1,\dots,m \text{ are locally Lipschitz continuous} \\ &\text{and of linear growth in the fourth variable,} \\ &\text{uniformly with respect to the first three variables.} \end{aligned} \tag{3.20}$$

We further assume that the functions are jointly measurable and adapted in the sense that for each j and $t \in [0,T]$, $g_j(t,\cdot)$ is $\mathscr{F}_t \otimes \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, where $\mathcal{B}_{[0,1]}$ and $\mathcal{B}_{\mathbb{R}}$ denote the sigma-algebras of the Borel sets on $[0,1]$ and $\mathbb{R}$, respectively.

The above assumptions on the coefficients on the edges, except for Assumption 3.10 which is specific for the graph setting, are analogous to those in [32, Section 5] and [33, Section 5].

To handle system (3.18), we rewrite it in the form of the abstract stochastic Cauchy-problem (SCP). To do so, we specify the functions appearing in (SCP) corresponding to (3.18).

The operator $(A, D(A)) = (A_p, D(A_p))$ will be the generator of the strongly continuous analytic semigroup $S := (T_p(t))_{t\geq 0}$ on the Banach space $E := E_p$ for some large $p \geq 2$, see Proposition 2.8 and Lemma 2.9. Hence, E is a UMD space of type 2.

For the function $F\colon [0,T] \times \Omega \times B \to B$ we have

$$F(t,\omega,u)(s) := (f_1(t,\omega,s,u_1(s)),\dots,f_m(t,\omega,s,u_m(s)))^\top, \qquad s \in [0,1]. \tag{3.21}$$

Since B is an algebra, Assumption 3.10 assures that F maps $[0,T] \times \Omega \times B$ into B.

To define the operator G we argue in analogy with [33, Section 5]. First define

$$H := E_2$$

the product L^2-space, see (2.3), which is a Hilbert space. We further define the multiplication operator $\Gamma\colon [0,T] \times B \to \mathcal{L}(H)$ as

$$[\Gamma(t,u)h]\,(s) := \begin{pmatrix} g_1(t,s,u_1(s)) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & g_m(t,s,u_m(s)) \end{pmatrix} \cdot \begin{pmatrix} h_1(s) \\ \vdots \\ h_m(s) \end{pmatrix}, \qquad s \in (0,1), \tag{3.22}$$

for $u \in B$, $h \in H$. Because of the assumptions (3.20) on the functions g_j, Γ clearly maps into $\mathcal{L}(H)$.

Let $(A_2, D(A_2))$ be the generator on $H = E_2$, see Proposition 2.5, and pick $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$. By (3.14) in the proof of Corollary 3.7 we have that there exists a continuous embedding

$$\imath\colon E_2^{\kappa_G} \to \left(\prod_{j=1}^{m} H_0^{2\kappa_G}(0,1;\mu_j dx)\right) \times \mathbb{R}^n =: \mathcal{H},$$

where $\mathcal{H}$ is a Hilbert space. Applying the steps (3.15) and (3.16) of Corollary 3.7 we obtain that $\mathcal{H} \hookrightarrow B$ holds, and by (3.11), there exists a continuous embedding

$$\jmath\colon \mathcal{H} \to E_p$$

for $p \geq 2$ arbitrary.

Define now G by

$$(-A_p)^{-\kappa_G} G(t,u)h := \jmath\,\imath\,(-A_2)^{-\kappa_G}\Gamma(t,u)h, \qquad u \in B,\ h \in H. \tag{3.23}$$

Proposition 3.12. *Let $p \geq 2$ and $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ be arbitrary. Then the operator G defined in (3.23) maps $[0,T] \times B$ into $\gamma(H, E_p^{-\kappa_G})$.*

Proof. We can argue as in [39, Section 10.2]. Using [39, Lemma 2.1(4)], we obtain in a similar way as in [39, Corollary 2.2]) that $\jmath \in \gamma(\mathcal{H}, E_p)$, since $2\kappa_G > \frac{1}{2}$ holds. Hence, by the definition of G and the ideal property of γ-radonifying operators, the mapping G takes values in $\gamma(H, E_p^{-\kappa_G})$. □

The driving noise process W_H is defined by

$$W_H(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_m(t) \end{pmatrix}, \qquad t \in [0,T], \tag{3.24}$$

and thus $(W_H(t))_{t\in[0,T]}$ is a cylindrical Wiener process, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, in the Hilbert space H with respect to the filtration $\mathbb{F}$.

We will state now the result regarding system (SCP) corresponding to (3.18).

Theorem 3.13. *Let F, G and W defined in (3.21), (3.23) and (3.24), respectively. Let $q > 4$ be arbitrary. Then for every $\xi \in L^q(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ a unique mild solution X of equation (SCP) exists globally and belongs to $L^q(\Omega; C([0,T]; B))$.*

Proof. The condition $q > 4$ allows us to choose $2 \le p < \infty$, $\theta \in [0, \frac{1}{2})$ and $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ such that

$$\theta > \frac{1}{2p} \tag{3.25}$$

and

$$0 \le \theta + \kappa_G < \frac{1}{2} - \frac{1}{q}.$$

We will apply Theorem 3.3 with θ and κ_G having the properties above. To this end we have to check Assumptions 3.2 for the mappings in (SCP), taking $A = A_p$ and $E = E_p$ for the p chosen above. Assumption (1) is satisfied because of the generator property of A_p, see Proposition 2.8. Assumption (2) is satisfied since (3.25) holds and we can use Corollary 3.7. Assumption (3) is satisfied by the statement of Proposition 3.8. Using that the functions f_j are polynomials of the 4th variable of the same degree $2k+1$ (see (3.19)), a similar computation as in [32, Example 4.2] and [32, Example 4.5], using techniques from [23, Section 4.3], shows that Assumptions (4) and (5) are satisfied for F with $N = m' = 2k+1$. By Proposition 3.12, G takes values in $\gamma(H, E_p^{-\kappa_G})$ with $H = E_2$ and κ_G chosen above. Using the assumptions (3.20) on the functions g_j and the proof of [39, Theorem 10.2], we obtain that G is locally Lipschitz continuous and of linear growth as a map $[0,T] \times B \to \gamma(H, E_p^{-\kappa_G})$, hence Assumption (6) holds. □

In the following theorem we will state a result regarding Hölder regularity of the mild solution of (SCP) corresponding to (3.18), see (3.3).

Theorem 3.14. *Let $q > 4$ be arbitrary, $\lambda, \eta > 0$ and $p \ge 2$ such that $\lambda + \eta > \frac{1}{2p}$. We assume that $\xi \in L^{(2k+1)q}(\Omega; E_p^{\lambda+\eta})$, where k is the constant appearing in (3.19). If the inequality*

$$\lambda + \eta < \frac{1}{4} - \frac{1}{q} \tag{3.26}$$

is fulfilled, then the mild solution X of (SCP) from Theorem 3.13 satisfies

$$X \in L^q(\Omega; C^\lambda([0,T], E_p^\eta)).$$

Proof. Using the continuous embedding (3.11), we have that

$$\xi \in L^{(2k+1)q}(\Omega; B)$$

holds. Since $(2k+1)q > 4$, by Theorem 3.13 there exists a global mild solution

$$X \in L^{(2k+1)q}(\Omega; C([0,T], B)).$$

This solution satisfies the following implicit equation (see (3.3)):

$$X(t) = S(t)\xi + S * F(\cdot, X(\cdot))(t) + S \diamond G(\cdot, X(\cdot))(t), \tag{3.27}$$

where S denotes the semigroup generated by A_p on E_p, $*$ denotes the usual convolution, $\diamond$ denotes the stochastic convolution with respect to $\mathcal{W}$. In the following we have to estimate the $L^q(\Omega; C^\lambda([0,T], E_p^\eta))$-norm of X, and we will do this using the triangle-inequality in (3.27).

For the qth power of first term we have

$$\begin{aligned}\mathbb{E}\|S(\cdot)\xi\|^q_{C^\lambda([0,T],E_p^\eta)} &= \mathbb{E}\left(\sup_{t,s\in[0,T]}\frac{\|S(t)\xi-S(s)\xi\|_{E_p^\eta}}{|t-s|^\lambda}\right)^q\\ &\leq \mathbb{E}\left(\sup_{h\in[0,T]}\frac{\|S(h)\xi-\xi\|_{E_p^\eta}}{|h|^\lambda}\right)^q\\ &= \mathbb{E}\left(\sup_{h\in[0,T]}\frac{\|S(h)(-A_p)^\eta\xi-(-A_p)^\eta\xi\|_{E_p}}{|h|^\lambda}\right)^q. \qquad (3.28)\end{aligned}$$

By assumption, $(-A_p)^\eta\xi \in D((-A_p)^\lambda)$ holds. Applying [26, Proposition II.5.33] we obtain that $(-A_p)^\eta\xi$ lies in the Hölder space of order λ on E_p, denoted by C_p^λ. Hence,

$$\sup_{h\in[0,T]}\frac{\|S(h)(-A_p)^\eta\xi-(-A_p)^\eta\xi\|_{E_p}}{|h|^\lambda} = \|(-A_p)^\eta\xi\|_{\mathsf{F}_{p,\lambda}} < \infty,$$

where $\|\cdot\|_{\mathsf{F}_{p,\lambda}}$ denotes the Favard norm of order λ on E_p, see [26, Definition II.5.10]. Furthermore, because of the continuous inclusion $D((-A_p)^\lambda) \hookrightarrow C_p^\lambda$, we have that there exists $c = c(\lambda)$ such that

$$\|(-A_p)^\eta\xi\|_{\mathsf{F}_{p,\lambda}} \leq c\cdot\|(-A_p)^\eta\xi\|_{E_p^\lambda} = c\cdot\|(-A_p)^{\lambda+\eta}\xi\|_{E_p}.$$

Hence,

$$\mathbb{E}\|S(\cdot)\xi\|^q_{C^\lambda([0,T],E_p^\eta)} \leq c\cdot\mathbb{E}\|(-A_p)^{\lambda+\eta}\xi\|^q_{E_p} < \infty$$

by assumption.

To estimate the qth power of the second term

$$\mathbb{E}\|S*F(\cdot,X(\cdot))\|^q_{C^\lambda([0,T],E_p^\eta)}$$

we choose $\theta > \frac{1}{2p}$ such that

$$\lambda+\eta+\theta < 1-\frac{1}{q}.$$

We will use [39, Lemma 3.6] with this θ, $\alpha = 1$, and q instead of p, and obtain that there exist constants $C \geq 0$ and $\varepsilon > 0$ such that

$$\|S*F(\cdot,X(\cdot))\|_{C^\lambda([0,T],E_p^\eta)} \leq CT^\varepsilon\|F(\cdot,X(\cdot))\|_{L^q(0,T;E_p^{-\theta})}. \qquad (3.29)$$

We have to estimate the expectation of the qth power on the right-hand-side of (3.29). By Corollary 3.7 we obtain

$$B \hookrightarrow E_p \hookrightarrow E_p^{-\theta},$$

since $\theta > \frac{1}{2p}$ holds and $(\omega'-A_p)^{-\theta}$ is an isomorphism between $E_p^{-\theta}$ and E_p. Using this and Assumptions 3.2(5) with $m' = 2k+1$ (which holds by the proof of Theorem 3.13), we have

$$\begin{aligned}\mathbb{E}\|F(\cdot,X(\cdot))\|^q_{L^q(0,T;E_p^{-\theta})} &= \mathbb{E}\int_0^T \|F(s,X(s))\|^q_{E_p^{-\theta}}\,ds\\ &\lesssim \mathbb{E}\int_0^T \|F(s,X(s))\|^q_B\,ds\\ &\lesssim \mathbb{E}\int_0^T \left(1+\|X(s)\|_B^{(2k+1)q}\right)ds\\ &\lesssim 1+\mathbb{E}\sup_{t\in[0,T]}\|X(t)\|_B^{(2k+1)q},\end{aligned}$$

where $\lesssim$ denotes that the expression on the left-hand-side is less or equal to a constant times the expression on the right-hand-side. This implies that for each $T > 0$ there exists $C_T > 0$ such that

$$\left(\mathbb{E}\|S * F(\cdot, X(\cdot))\|^q_{C^\lambda([0,T],E^\eta_p)}\right)^{\frac{1}{q}} \leq C_T \cdot \left(1 + \|X(t)\|^{2k+1}_{L^{(2k+1)q}(\Omega;C([0,T],B))}\right), \tag{3.30}$$

and the right-hand-side is finite.

To estimate the stochastic convolution term in (3.27) we first fix $0 < \alpha < \frac{1}{2}$ such that

$$\lambda + \eta + \frac{1}{4} < \alpha - \frac{1}{q}$$

holds. We now choose $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ such that

$$\lambda + \eta + \kappa_G < \alpha - \frac{1}{q}$$

is satisfied. Applying [39, Proposition 4.2] with $\theta = \kappa_G$ and q instead of p, we have that there exist $\varepsilon > 0$ and $C \geq 0$ such that

$$\mathbb{E}\,\|S \diamond G(\cdot, X(\cdot))\|^q_{C^\lambda([0,T],E^\eta_p)} \leq C^q T^{\varepsilon q} \int_0^T \mathbb{E}\,\|s \mapsto (t-s)^{-\alpha} G(s, X(s))\|^q_{\gamma(L^2(0,t;H),E_p^{-\kappa_G})}\, dt.$$

In the following we proceed similarly as done in the proof of [32, Theorem 4.3], with $N = 1$ and q instead of p. Since $E_p^{-\kappa_G}$ is a Banach space of type 2 (because E_p is of that type), the continuous embedding

$$L^2(0,t;\gamma(H, E_p^{-\kappa_G})) \hookrightarrow \gamma(L^2(0,t;H), E_p^{-\kappa_G})$$

holds. Using this, Young's inequality and the properties of G, respectively, we obtain the following estimates

$$\begin{aligned}
\mathbb{E}\,\|S \diamond G(\cdot, X(\cdot))\|^q_{C^\lambda([0,T],E^\eta_p)} &\lesssim T^{\varepsilon q} \int_0^T \mathbb{E}\,\|s \mapsto (t-s)^{-\alpha} G(s, X(s))\|^q_{L^2(0,t;\gamma(H,E_p^{-\kappa_G}))}\, dt \\
&= T^{\varepsilon q} \mathbb{E} \int_0^T \left(\int_0^t (t-s)^{-2\alpha}\, \|G(s, X(s))\|^2_{\gamma(H,E_p^{-\kappa_G})}\, ds\right)^{\frac{q}{2}} dt \\
&\leq T^{\varepsilon q} \left(\int_0^T t^{-2\alpha}\, dt\right)^{\frac{q}{2}} \mathbb{E} \int_0^T \|G(t, X(t))\|^q_{\gamma(H,E_p^{-\kappa_G})}\, dt \\
&\leq T^{(\frac{1}{2}-\alpha+\varepsilon)q} (c')^q \cdot \mathbb{E} \int_0^T (1 + \|X(t)\|_B)^q\, dt \\
&\lesssim T^{(\frac{1}{2}-\alpha+\varepsilon)q+1} (c')^q \cdot \left(1 + \mathbb{E}\|X(t)\|^q_{C([0,T],B)}\right).
\end{aligned}$$

Hence, for each $T > 0$ there exists constant $C'_T > 0$ such that

$$\left(\mathbb{E}\,\|S \diamond G(\cdot, X(\cdot))\|^q_{C^\lambda([0,T],E^\eta_p)}\right)^{\frac{1}{q}} \leq C'_T \cdot \left(1 + \|X(t)\|_{L^{(2k+1)q}(\Omega;C([0,T],B))}\right)^{2k+1}. \tag{3.31}$$

In summary, by (3.28), (3.30) and (3.31), we obtain that $X \in L^q(\Omega; C^\lambda([0,T], E^\eta_p))$ holds, hence the proof is completed. $\square$

We are now in the position to finally consider (1.1). Let

$$\beta := \max_{1\le j\le m} \beta_j.$$

We also introduce

$$f_j(\eta) := f(\eta) = -\eta^3 + \beta^2\eta. \tag{3.32}$$

and

$$\varrho_j := \beta^2 - \beta_j^2 \ge 0.$$

With these notations, we can rewrite (1.1) in an equivalent form as

$$\begin{cases} \dot u_j(t,x) = (c_j u_j')'(t,x) - \tilde p_j(x)u_j(t,x) & \\ \qquad\qquad + f_j(u_j(t,x)) & \\ \qquad\qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t\in(0,T],\ x\in(0,1),\ j=1,\dots,m,\\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t\in(0,T],\ \forall j,\ell\in\Gamma(\mathsf{v}_i),\ i=1,\dots,n,\\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t\in(0,T],\ i=1,\dots,n,\\ u_j(0,x) = \mathsf{u}_j(x), & x\in[0,1],\ j=1,\dots,m, \end{cases} \tag{3.33}$$

with $\tilde p_j(x) := p_j(x) + \varrho_j$, $j = 1,\dots m$.

We define the operator A_p on E_p as in (2.5) with $\tilde p_j$'s instead of p_j's and with domain (2.11).

Theorem 3.15. *Let F, G and W defined in (3.21), (3.23) and (3.24), respectively, for the system (3.33). Let $q > 4$ be arbitrary. Then for every $\xi \in L^q(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ a unique mild solution X of equation (SCP) corresponding to (3.33), which is equivalent to (1.1), exists globally and belongs to $L^q(\Omega; C([0,T]; B))$. Let $\lambda, \eta > 0$, $p \ge 2$ be arbitrary constants such that $\lambda + \eta > \frac{1}{2p}$. If $\xi \in L^{3q}(\Omega; E_p^{\lambda+\eta})$ and the inequality*

$$\lambda + \eta < \frac{1}{4} - \frac{1}{q}$$

is fulfilled, then $X \in L^q(\Omega; C^\lambda([0,T], E_p^\eta))$.

Proof. First note that the coefficients $\tilde p_j$ stay nonnegative as the constants ϱ_j are nonnegative. Furthermore, the nonlinear terms $f_j = f$ in (3.32) are of the form (3.19) with $k = 1$ and constant coefficients. Hence, Assumption 3.10 is fulfilled by Remark 3.11. The statement then follows from Theorems 3.13 and 3.14. □

3.4 Concluding remarks

In equation (3.18a) we could have prescribed coloured noise instead of white noise on the edges of the graph. That is, we could set

$$\begin{aligned} \dot u_j(t,x) &= (c_j u_j')'(t,x) - p_j(x)u_j(t,x) \\ &\quad + f_j(t,x,u_j(t,x)) \\ &\quad + g_j(t,x,u_j(t,x))R_j\frac{\partial w_j}{\partial t}(t,x), \qquad t\in(0,T],\ x\in(0,1),\ j=1,\dots,m, \end{aligned} \tag{3.34}$$

with $R_j \in \gamma(L^2(0,1;\mu_j dx), L^p(0,1;\mu_j dx))$. Then we define

$$R := \begin{pmatrix} R_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & R_m \end{pmatrix} \in \gamma(H, E_p)$$

with $H = E_2$ and $p \geq 2$ arbitrary. Using this, we can define the operator $G : [0,T] \times B \to \gamma(H, E_p)$ as

$$G(t,u)h := \Gamma(t,u)Rh, \quad h \in H,$$

where the operator $\Gamma : [0,T] \times B \to \mathcal{L}(H)$ is defined in (3.22). It is easy to see that G satisfies Assumptions 3.2(6) with $\kappa_G = 0$. For example, if $u, v \in B$ with $\|u\|, \|v\| \leq r$, then

$$\begin{aligned} \|G(t,u) - G(t,v)\|_{\gamma(H,E_p)} &\leq \|\Gamma(t,u) - \Gamma(t,v)\|_{\mathcal{L}(E_p)} \cdot \|R\|_{\gamma(H,E_p)} \\ &\leq L^{(r)} \cdot \|u - v\|_B \cdot \|R\|_{\gamma(H,E_p)} \end{aligned}$$

where $L^{(r)}$ is the maximum of the Lipschitz-constants of the functions g_j on the ball of radius r.

If setting (3.34) instead of (3.18a), Theorem 3.13 remains true as stated; that is, for $q > 4$, but one may use a simpler Hilbert space machinery; that is, one may set $p = 2$ in the proof. However, in the coloured noise case, Theorem 3.13 is true also for $q > 2$. But this can only be shown by choosing $p > 2$ large enough in the proof and hence, in this case, the Banach space arguments are crucial.

In Theorem 3.14, if one takes $p = 2$ (Hilbert space) and $q > 4$, then the statement is true for $\lambda + \eta > \frac{1}{4}$ with

$$\lambda + \eta < \frac{1}{2} - \frac{1}{q} \tag{3.35}$$

instead of (3.26). In this case R will be a Hilbert–Schmidt operator whence the covariance operator of the driving process is trace-class. However, the statement of the theorem remains true for $q > 2$ as well assuming (3.35) instead of (3.26), but only for the Banach space E_p for p large enough so that $\lambda + \eta > \frac{1}{2p}$.

The statements of Theorem 3.15 could also be changed accordingly.

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